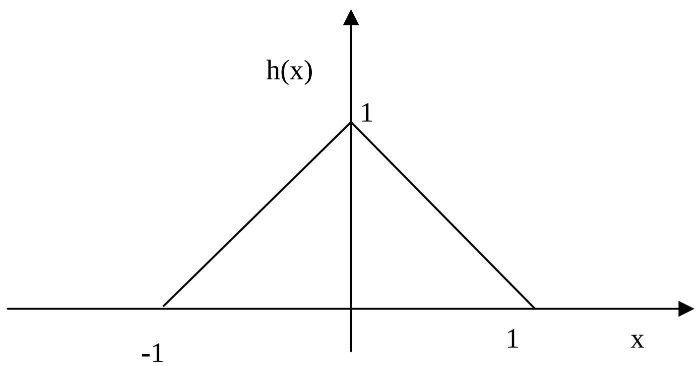
SD575 Digital Image Processing – Midterm Examination
October 18, 2007 8:45am – 10:15am
Prof. D. A. Clausi

1. Consider an LSI system with impulse response h(x):



(a) Determine the response g(x) of the system to a unit step. What is the impact (qualitatively) on the image when h(x) is applied?
(b) Is the system h(x) causal? Explain. Is the system h(x) memoryless? Explain.
(c) Find the frequency response $H_1(u)$ that compensates for h(x).
(d) For low frequencies, $H_1(u)$ can be approximated with the following form of the frequency response:

$$\mathbf{H_1(u)} = \frac{1}{1 - c^2u^2}$$

Find c. Hint: Use the Taylor series expansion $\cos x \sim 1 - 0.5\, x^2$
(e) If f(x) and g(x) are the input and output of the system of H(u), what is the differential equation relating f(x) and g(x)?

2. Consider the following probability density function that represents the probability of occurrence for grey levels in an image:

$$p_f(r) = k\, e^{-ar} \qquad 0 \le r \le 1$$

(a) Derive k to ensure that $p_f(r)$ is a true pdf.
(b) What point operation maps this pdf to a uniform distribution? How is the local contrast affected in both light and dark regions of the input image?

Now consider the discrete histogram $\mathbf{H(k) = 128(0.5)^k,\ 0 \le k \le 7}$

(c) Devise a histogram equalization solution, provide the transformation, and sketch your resulting histogram.
(d) Is histogram equalization in general a linear system? an invertible system? a memoryless system? For each case, explain your answer.

$$H_{inv}(u) = \left(\frac{\pi u}{\sin \pi u}\right)^2$$

(d) $$H_{inv}(u) = \frac{\pi^2 u^2}{1 - \cos^2 \pi u} \approx \frac{\pi^2 u^2}{1 - (1 - \frac{1}{2}\pi^2 u^2)^2}$$

$$= \frac{\pi^2 u^2}{\pi^2 u^2 - \frac{1}{4}\pi^4 u^4}$$

$\therefore$ for small u, $H_{inv}(u) \approx \dfrac{1}{1 - \frac{1}{4}\pi^2 u^2}$ and $c = \dfrac{\pi}{2}$

(e) $$H(u) = \frac{G(u)}{F(u)} = 1 - \frac{1}{4}\pi^2 u^2$$ (for low frequencies)

$$\therefore g(x) = f(x) + \frac{1}{16} f''(x)$$

Brute Force (Q 1(c))

$$H(u) = \int_{-\infty}^{\infty} h(x)\, e^{-j2\pi ux}\, dx = \int_{-1}^{0} (x+1)\, e^{-j2\pi ux}\, dx + \int_{0}^{1} (-x+1)\, e^{-j2\pi ux}\, dx$$

$$= \underbrace{\int_{-1}^{0} x e^{-j2\pi ux}\, dx}_{A} + \underbrace{\int_{-1}^{0} e^{-j2\pi ux}\, dx}_{B} - \underbrace{\int_{0}^{1} x e^{-j2\pi ux}\, dx}_{C} + \underbrace{\int_{0}^{1} e^{-j2\pi ux}\, dx}_{D}$$

A $\int_{-1}^{0} x e^{-j2\pi ux}\, dx$

$u = x$, $du = dx$, $v = \dfrac{e^{-j2\pi ux}}{-j2\pi u}$, $dv = e^{-j2\pi ux}$

$$\int u\, dv = uv - \int v\, du$$

$$= \frac{xe^{-j2\pi ux}}{-j2\pi u}\Bigg|_{-1}^{0} + \int_{-1}^{0} \frac{e^{-j2\pi ux}}{j2\pi u}\, dx$$

$$= -\frac{e^{-j2\pi u}}{j2\pi u} + \frac{e^{-j2\pi ux}}{4\pi^2 u^2}\Bigg|_{-1}^{0}$$

$$= \cancel{\frac{-e^{+j2\pi u}}{j2\pi u}} + \frac{1}{4\pi^2 u^2} - \frac{e^{j2\pi u}}{4\pi^2 u^2}$$

B $\int_{-1}^{0} e^{-j2\pi ux}\, dx = \cancel{\dfrac{e^{+j2\pi u}}{j2\pi u}} - \cancel{\dfrac{1}{j2\pi u}}$

C $-\int_{0}^{1} x e^{-j2\pi ux}\, dx = -\left[\dfrac{xe^{-j2\pi ux}}{-j2\pi u}\Bigg|_{0}^{1} + \dfrac{e^{-j2\pi ux}}{4\pi^2 u^2}\Bigg|_{0}^{1} \right]$

$$= \cancel{\frac{e^{-j2\pi u}}{j2\pi u}} - \frac{e^{-j2\pi u}}{4\pi^2 u^2} + \frac{1}{4\pi^2 u^2}$$

D $\int_{0}^{1} e^{-j2\pi ux}\, dx = \cancel{\dfrac{1}{j2\pi u}} - \cancel{\dfrac{e^{-j2\pi u}}{j2\pi u}}$

A + B + C + D $= \dfrac{2}{4\pi^2 u^2} - \dfrac{e^{j2\pi u}}{4\pi^2 u^2} - \dfrac{e^{-j2\pi u}}{4\pi^2 u^2}$

$$= \left(\frac{e^{j\pi u} - e^{-j\pi u}}{j2\pi u} \right)^2$$

$$= \left(\frac{\sin \pi u}{\pi u} \right)^2$$

2. (a) $k\int_0^1 e^{-ar}\,dr = 1$

$$k\left[\frac{e^{-ar}}{-a}\right]_0^1 = 1$$

$$k\left[\frac{e^{-a}-1}{-a}\right] = 1 \implies k = \frac{a}{1-e^{-a}}$$

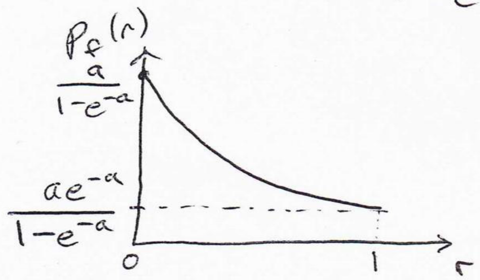
$$\therefore\ p_r(r) = \frac{a}{1-e^{-a}}e^{-ar} \qquad 0 \le r \le 1$$

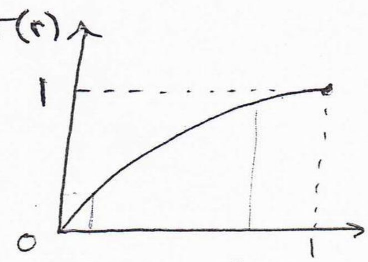
(b) $$T(r) = \int_0^r \frac{a}{1-e^{-a}}e^{-a\alpha}\,d\alpha \qquad 0 \le r \le 1$$

$$= \frac{a}{1-e^{-a}}\left[\frac{e^{-a\alpha}}{-a}\right]_0^r$$

$$= \frac{\cancel{a}}{1-e^{-a}}\left[\frac{e^{-ar}-1}{-\cancel{a}}\right]$$

$$= \frac{1-e^{-ar}}{1-e^{-a}}, \quad 0 \le r \le 1$$





stretch slope>1

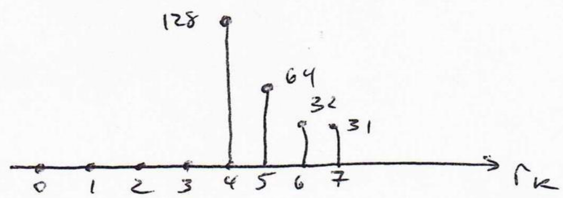
compress slope<1

$$128\sum_{k=0}^{7}(1/2)^k = 128\left[\frac{1-(1/2)^8}{1-1/2}\right] = 256\left(\frac{256-\cancel{128}\,1}{256}\right) = 255$$ M-5

(c) $H(k) = 128\,(1/2)^k \qquad 0 \le r_k \le 7$

r_k	0	1	2	3	4	5	6	7
$p_r(r_k)$	$\frac{128}{255}$	$\frac{64}{255}$	$\frac{32}{255}$	$\frac{16}{255}$	$\frac{8}{255}$	$\frac{4}{255}$	$\frac{2}{255}$	$\frac{1}{255}$
cdf	$\frac{128}{255}$	$\frac{192}{255}$	$\frac{224}{255}$	$\frac{240}{255}$	$\frac{248}{255}$	$\frac{252}{255}$	$\frac{254}{255}$	$\frac{255}{255}$
$(L-1)$cdf	$\frac{896}{255}$	$\frac{1344}{255}$	$\frac{1568}{255}$	$\frac{1680}{255}$	$\frac{1736}{255}$	$\frac{1764}{255}$	$\frac{1778}{255}$	7
round(s_k)	4	5	6	7	7	7	7	7
output	0	0	0	0	128	64	32	31

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j)$$



(d) linear ⇒ NO

- generally mapping fn is non-linear

invertible ⇒ YES, if continuous

NO, if discrete due to bin filling

memoryless ⇒ YES, point operation